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A Bijection for Partitions with All Ranks at Least t

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It follows from the work of Andrews and Bressoud that for $t \leqslant 1$, the number of partitions of n with all successive ranks at least t is equal to the number of partitions of n with no part of size $2-t$. We give a simple bijection for this identity which generalizes a result of Cheema and Gordon for 2-rowed plane partitions. The bijection yields several refinements of the identity when the partition counts are parametrized by the number of parts and/or the size of the Durfee rectangle. In addition, it gives an interpretation of the difference of (shifted) successive Gaussian polynomials which we relate to other interpretations of Andrews and Fishel.

1. INTRODUCTION

A partition π of a non-negative integer n is a nonincreasing sequence $\pi = (\pi_1, ..., \pi_t)$ of positive integers whose sum is n and the *weight* of π, denoted $|\pi|$, is n. The Ferrers diagram of π is an array of dots, left justified, in which the number of dots in row i is π_i. The largest square subarray of dots in this diagram is the Durfee square and $d(\pi)$ refers to the length of a side. The conjugate of π, denoted π', is the partition whose ith part is the

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number of dots in the ith column of the Ferrers diagram of π. The *sequence of successive ranks* of π is the sequence $(\pi_1 - \pi'_1, ..., \pi_d - \pi'_d)$, where $d = d(\pi)$ [Dys44, Atk66].

Let $P(n)$ denote the set of all partitions of n and for integer $b > 0$, let $P_b(n)$ be the set of partitions of n with no part equal to b. If we split $P(n)$ into those partitions which do not contain a "b" ($P_b(n)$) and those which do contain a "b" (remove one "b") then it can be seen that:

$$|P_b(n)| = |P(n)| - |P(n-b)|. \tag{1}$$

We consider generalizations and refinements of the identity

$$|R(n)| = |P_1(n)|, \tag{2}$$

where $R(n)$ is the set of those partitions of n with all successive ranks positive. As observed in [ER93], (2) follows from Theorem 1 in Bressoud [Bre80] which is an extension of Theorem 5 in Andrews [And71] to even as well as odd moduli. The results of Bressoud and Andrews are actually a generalization of the Rogers–Ramanujan identities and (2) follows as a very special case. Direct proofs of (2) can be found in [And93] and [RA95], but apparently no simple bijective proof of the result has appeared. However, bijections are implicit in earlier work of Cheema and Gordon [CG64] and of Burge [Bur81], as we discuss in Sections 5 and 7.

The family $R(n)$ has received attention recently in connection with graphical partitions, that is, partitions which are the degree sequences of simple graphs [ER93, RA95, BS95]. It was pointed out by Erdős and Richmond in [ER93] that the conjugate of any partition in $R(n)$ is graphical and thus, in view of (1) and (2), the number of graphical partitions is at least $|R(n)| = |P(n)| - |P(n-1)|$, which is known to be asymptotically $\pi\, |P(n)|/\sqrt{6n}$ [RS54].

In Section 2 of this paper, we give a simple bijective proof of (2). The bijection is based on a result of Cheema and Gordon [CG64].

In Section 3, we consider a generalization of (2) which also follows from the Andrews–Bressoud theorem, stated below.

THEOREM 1 [And71, Bre80]. *For integers M, r, satisfying $0 < r < M/2$, the number $B_{M,r}(n)$ of partitions of n whose successive ranks lie in the interval $[-r+2, M-r-2]$ is equal to the number $A_{M,r}(n)$ of partitions of n with no part congruent to 0, r, or $-r$ modulo M.*

(For $r = 1$, $M = 4$ and $r = 2$, $M = 5$, this gives the Rogers–Ramanujan identities.)

Let $R_{\geqslant t}(n)$ denote the set of partitions of n in which all ranks are at least t. Then for $1-n \leqslant t \leqslant 1$, it follows from Theorem 1 by setting $r=2-t$ and $M=n+r+1$ that

$$|R_{\geqslant t}(n)| = |P_{2-t}(n)|. \tag{3}$$

In Section 3, we give a bijection for (3) by first showing bijectively that for $t \leqslant 0$,

$$|R_{=t}(n)| = |R_{\geqslant 1}(n-1+t)|, \tag{4}$$

where $R_{=t}(n)$ is the set of partitions whose minimum rank is exactly t, that is,

$$R_{=t}(n) = R_{\geqslant t}(n) - R_{\geqslant t+1}(n).$$

We define the *Durfee rectangle* of a partition π to be the largest $d \times (d+1)$ rectangle contained in the Ferrers diagram of π and let $d^*(\pi)$ denote the height of the Durfee rectangle of π.

It turns out that the bijections in Sections 2 and 3 which establish (2) and (4) preserve both the number of parts in a partition and the size of the Durfee rectangle. (The size of the Durfee square need not be preserved!) As a result, we get several refinements of identities (2), (3), and (4) which are highlighted in Section 4 in terms of generating functions. In Section 5, we note the connection with plane partitions and a result of Cheema and Gordon when $t \geqslant 1$. Our bijection gives an interpretation of the difference of successive Gaussian polynomials which we relate to other interpretations in Section 6. In Section 7, we describe a result of Burge [Bur81] from which (3) also follows as a special case. We note that although the result of Burge is proved bijectively, his bijection, for the special case of (3), is not the same as ours and does not preserve the statistics we require for the refinements in Section 4.

2. A BIJECTION FOR PARTITIONS WITH ALL RANKS POSITIVE

In this section, we give a simple bijection between $P_1(n)$ and $R_{\geqslant 1}(n)$ which we derived from a mapping of Cheema and Gordon as described in Section 5. Define the *rank vector* of a partition π to be the vector $[r_1(\pi), r_2(\pi), ..., r_{d(\pi)}(\pi)]$ whose ith entry is the ith successive rank $\pi_i - \pi'_i$ of π.

Define a partial function F on all partitions π as follows. $F(\pi)$ is that partition obtained from π by the following procedure:

$F(\pi)$:

While some rank of π is less than 1 do the following:

1. Let t be the minimum rank of π.
2. Let i be the largest index such that $r_i(\pi) = t$.
3. Delete a part of size i from π.
4. Add a part of size $i-1$ to π'.
5. Add a part of size 1 to π.

So, for example, $F((6, 5, 4, 4, 4, 3, 2, 2)) = ((10, 8, 6, 2, 1, 1, 1, 1))$, as illustrated in Fig. 1, and $F((6, 5, 4, 1)) = ((6, 5, 4, 1))$. However,

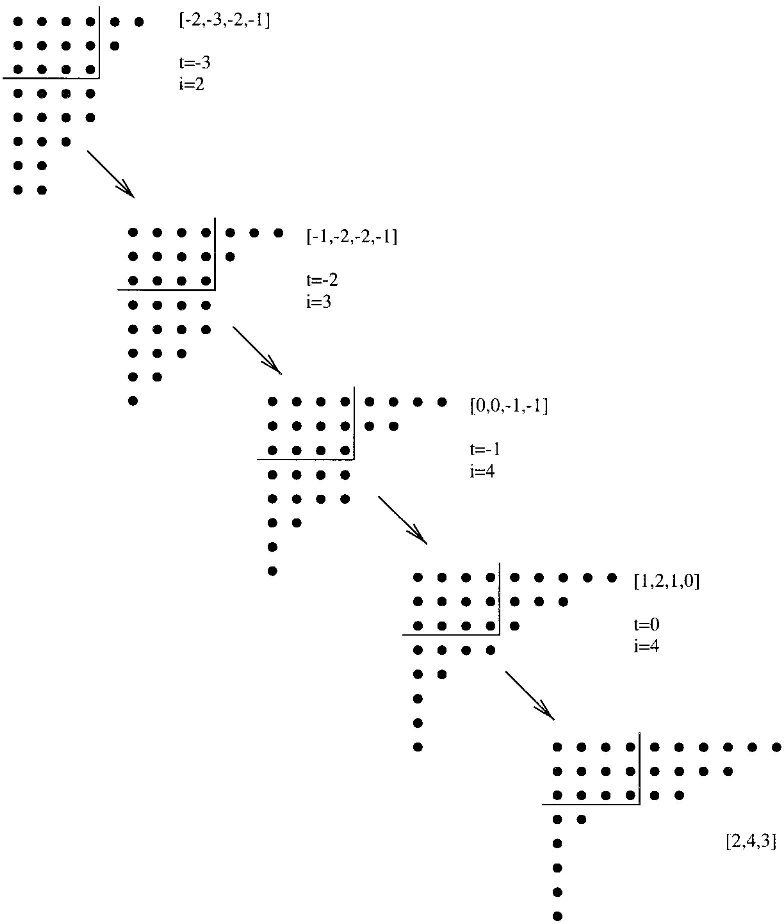

FIG. 1. Computation of $F((6, 5, 4, 4, 4, 3, 2, 2))$, with rank vector shown at each iteration and with Durfee rectangle indicated.

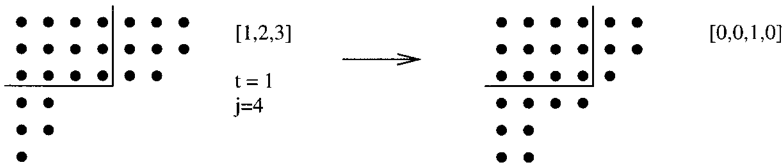

FIG. 2. Computation of $G((7, 7, 6, 2, 2, 1))$, which uses case b(ii) of G.

$F((4, 4, 3, 2, 1))$ is undefined since the procedure does not terminate. F would also be undefined if it happened that at step (3), π contained no part of size i.

Define another partial function G on partitions π so that $G(\pi)$ is that partition obtained from π by the following procedure.

$G(\pi)$:

While π contains a part of size 1 do the following:

a. Let t be the minimum rank of π.

b. (i) If $t > 1$, let $j = d(\pi) + 1$.

(ii) Otherwise, if $r_i(\pi) = t$ only for $i = 1$, let $j = d(\pi) + 1$.

(iii) Otherwise, let j be the smallest index with $j > 1$ and $r_j(\pi) = t$.

c. Delete a part of size $j - 1$ from π'.

d. Add a part of size j to π.

e. Delete a part of size 1 from π.

For example, then $((10, 8, 6, 2, 1, 1, 1, 1)) = ((6, 5, 4, 4, 4, 3, 2, 2))$ (read Fig. 1 in reverse.) Note that in the first iteration of G for this example, case b(i) applies. Examples of b(ii) and b(iii) occur in the first iteration of the computations of $G((7, 7, 6, 2, 2, 1)) = (6, 6, 5, 4, 2, 2))$ (Fig. 2) and $G((8, 6, 5, 2, 2, 1, 1)) = (6, 6, 5, 2, 2, 2, 2))$ (Fig. 3), respectively. Partition $(3, 2, 2)$ is fixed by G. G would be undefined if in step (c), π' contained no part of size $j - 1$.

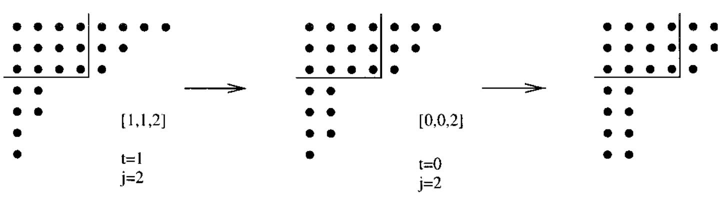

FIG. 3. Computation of $G((8, 6, 5, 2, 2, 1, 1))$, which uses case b(iii) of G.

Our main result is:

THEOREM 2. *F gives a bijection, with inverse G, from the set of partitions with no part equal to 1 to the set of partitions with all ranks positive. Furthermore, when applied to a partition π with no "1," F preserves the weight of π, the number of parts of π, and the size of the Durfee rectangle of π.*

Figure 4 shows the one-to-one correspondence F: $P_1(10) \to R_{\geqslant 1}(10)$.

To prove Theorem 2, we first give conditions which guarantee that in step (3) of procedure F, π will have a part of size i (Lemma 1 below) and that in step (c) of procedure G, π' will have a part of size $j-1$ (Lemma 2 below).

LEMMA 1. *Let t be the minimum rank of π and let i be the largest index with $r_i(\pi)=t$. If $t \leqslant 0$, then π contains a part of size i.*

Proof. Let $d=d(\pi)$. By definition of i, $i \leqslant d$, so both π and π' contain parts of size at least i. If $i=d$, then $r_d=\pi_d-\pi'_d=t \leqslant 0$. Thus $\pi'_d=\pi_d-t \geqslant d-t$ and so $\pi_{d-t}=d=i$. Otherwise $(i<d)$, let j be the largest index with $\pi_j \geqslant i$. If $\pi_j \geqslant i+1$, then $\pi'_{i+1}=j=\pi'_i$. But then since $\pi_i \geqslant \pi_{i+1}$, we would have

$$r_i=\pi_i-\pi'_i \geqslant \pi_{i+1}-\pi'_{i+1}=r_{i+1},$$

contradicting choice of i. ∎

$P_1(10)$		$R_{\geq 1}(10)$
(10)	$\longrightarrow$	(10)
(8,2)	$\longrightarrow$	(9,1)
(7,3)	$\longrightarrow$	(7,3)
(6,4)	$\longrightarrow$	(6,4)
(6,2,2)	$\longrightarrow$	(8,1,1)
(5,5)	$\longrightarrow$	(5,5)
(5,3,2)	$\longrightarrow$	(6,3,1)
(4,4,2)	$\longrightarrow$	(4,4,2)
(4,3,3)	$\longrightarrow$	(5,4,1)
(4,2,2,2)	$\longrightarrow$	(7,1,1,1)
(3,3,2,2)	$\longrightarrow$	(5,3,1,1)
(2,2,2,2,2)	$\longrightarrow$	(6,1,1,1,1)

FIG. 4. The bijection F: $P_1(10) \to R_{\geqslant 1}(10)$.

LEMMA 2. *Let t be the minimum rank of π and let $d = d(\pi)$.*

(i) *If $t > 1$ then π' contains a part of size d.*

(ii) *If $t = 1$ and if $r_i = t$ only for $i = 1$, then π' contains a part of size d.*

(iii) *If $t \leqslant 1$ and $r_i = t$ for some $i > 1$, let j be the smallest such index. If π contains a part of size 1, then π' contains a part of size $j - 1$.*

Proof. In cases (i) and (ii), $r_d \geqslant 1$, so $\pi_d > d$ and therefore $\pi'_{d+1} = d$. In case (iii), it suffices to show that $\pi_{j-1} > \pi_j$. If $j = 2$, $\pi_1 > \pi_2$ since π contains a 1. Otherwise, $j \geqslant 3$ and by definition of t and j, $r_{j-1} > r_j$ and therefore $\pi_{j-1} > \pi'_{j-1} + (\pi_j - \pi'_j) \geqslant \pi_j$. ∎

We now focus on the effect of one iteration of steps (1)–(5) of the computation of $F(\pi)$. Let f be the partial function which assigns to a partition π the partition $f(\pi)$ derived from π by one application of steps (1)–(5) of procedure $F(\pi)$. Similarly, let g be the partial function which maps π to the partition resulting from one application of steps (a)–(e) of procedure $G(\pi)$. Then, e.g., $f((6, 5, 4, 4, 4, 3, 2, 2)) = (7, 5, 4, 4, 4, 3, 2, 1)$ and $f((7, 5, 4, 4, 4, 3, 2, 1)) = (8, 6, 4, 4, 4, 2, 1, 1)$ (see Fig. 1); f fixes both $(6, 5, 4, 1)$ and $(4, 4, 3, 2, 1)$. Also, e.g., $g((7, 7, 6, 2, 2, 1)) = (6, 6, 5, 4, 2, 2)$ (Fig. 2); $g((8, 6, 5, 2, 2, 1, 1)) = (7, 6, 5, 2, 2, 2, 1)$ (Fig. 3); g fixes $(3, 2, 2)$.

Because we use these repeatedly in the following proofs, define $i(\pi)$, $j(\pi)$ to be the values assigned to i and j by application of f, g, respectively, to π.

LEMMA 3. *For $s \geqslant 0$, f gives a bijection, with inverse g:*

$$f\colon \{\pi \in R_{=0}(n) \mid i(\pi) > 1, \pi \text{ has } s \text{ ones}\} \to \{\pi \in R_{\geqslant 1}(n) \mid \pi \text{ has } s+1 \text{ ones}\},$$

and, if $t < 0$,

$$f\colon \{\pi \in R_{=t}(n) \mid i(\pi) > 1, \pi \text{ has } s \text{ ones}\}$$
$$\to \{\pi \in R_{=t+1}(n) \mid i(\pi) > 1, \pi \text{ has } s+1 \text{ ones}\}.$$

Furthermore, for π in these domains, f preserves the number of parts, the size of the Durfee rectangle, and, if $t < 0$, the size of the Durfee square; f increases the size of the largest part by 1.

Proof. For $t \leqslant 0$, let $\pi \in R_{=t}(n)$ with $i = i(\pi) > 1$. By Lemma 1, π has a part of size $i \leqslant d = d(\pi)$. Let k be the largest index such that $\pi_k = i$, i.e., π_k is the last occurrence of i in π.

Case $k>d$. If $k>d$, then steps (3)–(5) of f convert the rank vector $[r_1, ..., r_d]$ of π to

$$[r_1+1, r_2+2, ..., r_{i-1}+2, r_i+1, r_{i+1}, ..., r_d]$$

and the smallest rank is now $t+1=r_i+1$, which occurs at position i, but can also occur at any of the positions $\{1, i+1, ..., d\}$. Thus, $i(f(\pi)) \geqslant i > 1$ and

$$f(\pi) \in \{\sigma \in R_{=t+1}(n) \mid i(\sigma) > 1\}.$$

Clearly, $f(\pi)$ has the same number of parts as π and one more 1 than π and, since $k>d$ and $i(\pi) \leqslant d$, the same size Durfee square and rectangle as π. The largest part of $f(\pi)$ is one more than the largest part of π. To compute $g(f(\pi))$, note that the minimum rank of $f(\pi)$ is $t+1 \leqslant 1$ and j is chosen as in step b(iii) of g to be just $i(\pi) \geqslant 1$. By Lemma 2(iii), the conjugate of $f(\pi)$ has a part of size $j-1$, so $g(f(\pi))=\pi$.

Case $k=d$. If $k=d$, then $\pi_d=d=i$ and $\pi_{d+1}<d$, so $r_d=0$, the minimum rank of π is 0 and the Durfee rectangle of π is $(d-1)\times d$. Applying steps (3)–(5) of f to π increases the number of parts "1" by one and decreases the Durfee square size by 1, but not the size of the Durfee rectangle. The rank vector $[r_1, ..., r_d]$ of π becomes $[r_1+1, r_2+2, ..., r_{d-1}+2]$. If r_1 was 0, the new minimum rank is 1 and it occurs only at position 1. In this case, in computing $g(f(\pi))$, case b(ii) of g applies and steps (c)–(e) of g send $f(\pi)$ to π. Otherwise, the minimum rank can occur at any position and have any value larger than 1, in which case g will send $f(\pi)$ to π since case b(i) of g applies. Note that in both of these cases, by Lemma 2(i) and (ii), the conjugate of $f(\pi)$ has a part of size $d-1$. Either way, $f(\pi) \in R_{\geqslant 1}(n)$ and $f(\pi)$ has the same number of parts as π and the size of the largest part has increased by 1.

To show that f is onto, consider first the case when $t<0$. Let τ be a partition in $R_{=t+1}(n)$ with $s+1$ ones and $i(\tau)>1$. Then j will be chosen as in step b(iii) of g. So $0<j\leqslant d$ and steps (c)–(d) of g delete a "1" from τ but do not change the size of the Durfee square or Durfee rectangle of τ. Lemma 2(iii) guarantees that τ has a part of size $j-1$. Then if τ has rank vector $[r_1, ..., r_d]$, $g(\tau)$ has rank vector

$$[r_1-1, r_2-2, ..., r_{j-1}-2, r_j-1, r_{j+1}, ..., r_d].$$

So, by definition of j, the minimum rank of $g(\tau)$ is $r_j-1=(t+1)-1=t$, which can occur at any of the positions 1, 2, ..., j, but the last occurrence is at position j. Thus $i(g(\tau))=j>1$ and $f(g(\tau))=\tau$.

Continuing the proof that f is onto, when $t=0$, if $\tau \in R_{\geqslant 1}(n)$, with $s+1 \geqslant 1$ parts "1," let t' be the minimum rank of τ. If $t'>1$ or if $t'=1$ and rank t' occurs only at $r_1(\tau)$, then step b(i) or b(ii) of g sets j to $d+1$, where now $d=d(\tau)$. In this case, $r_d>1$, so $\tau_d \geqslant d+2$ but $\tau_{d+1} \leqslant d$. In particular, τ has Durfee rectangle of size $d \times (d+1)$. After steps (c)–(d) of g, τ loses a "1" and the $d+1$-st part of $g(\tau)$ will be $d+1$, so $g(\tau)$ will have Durfee square size $d+1$, but Durfee rectangle still $d \times (d+1)$. If τ had rank vector $[r_1, ..., r_d]$, $g(\tau)$ will have rank vector $[r_1-1, r_2, -2, ..., r_d-2, 0]$. Since all ranks of τ were greater than 1, except possibly $r_1=1$, $g(\tau)$ has minimum rank 0, whose last occurrence is at location $d+1$. Thus $i(g(\tau))>0$ and $f(g(\tau))=\tau$ in this case. Finally, if $t' \leqslant 1$ and $i(\tau)>1$ then j is chosen as in step b(iii). Then just as in the case $t<0$, it follows that $g(\tau)$ has minimum rank 0 and exactly s parts "1," $i(g(\tau))>1$ and $f(g(\tau))=\tau$. ▮

Proof of Theorem 2. For partition π of n with no part "1," let t be the minimum rank. If $t \geqslant 1$, then $F(\pi)=\pi$ and $G(\pi)=\pi$. Otherwise, since π has no "1," $\pi'_1=\pi'_2$ and therefore $r_1=\pi_1-\pi'_1 \geqslant \pi_2-\pi'_2=r_2$ so that $i(\pi)>1$. Thus by repeated application of Lemma 3, $F(\pi)=f^{1-t}(\pi)$ is a partition in $R_{\geqslant 1}(n)$ with exactly $1-t$ ones and $G(F(\pi))=g^{1-t}(f^{1-t}(\pi))=\pi$, showing that F is one-to-one. To show F is onto, if τ is a partition in $R_{\geqslant 1}(n)$ with exactly s parts of size 1, again by repeated application of Lemma 3, $G(\tau)=g^s(\tau)$ is a partition in $R_{=1-s}(n)$ with no "1" (and therefore with $i(g^s(\tau))>1$)and $F(G(\tau))=f^{1-(1-s))}g^s(\tau)=\tau$. ▮

3. THE BIJECTION FOR $R_{\geqslant t}(n)=P_{2-t}(n)$

Let f^* be the function defined by one application of steps (1)–(4) of F and let g^* be the following modification of steps (a)–(d) of G.

$g^*(\pi)$:

a. Let t be the minimum rank of π.

b. (i) If $t>1$, let $j=d(\pi)+1$.

(ii) Otherwise, let j be the smallest index with $r_j=t$.

c. Delete a part of size $j-1$ from π'.

d. Add a part of size j to π.

The proof of Lemma 3 can be modified to show the following.

LEMMA 4. *f^* gives a bijection, with inverse g^*:*

$$f^*\colon R_{=0}(n) \rightarrow R_{\geqslant 1}(n-1)$$

and, if $t<0$,

$$f^{*}\colon R_{=t}(n) \to R_{=t+1}(n-1)$$

Furthermore, for π *in these domains,* f^{*} *preserves the size of the Durfee rectangle, and, if* $t<0$, *the size of the Durfee square;* f^{*} *increases the size of the largest part by* 1 *if* $i(\pi)\neq 1$ *and decreases the number of parts by* 1.

It follows then by repeated application of Lemma 4 that for $t\leqslant 1$,

$$|R_{=t}(n)| = |R_{\geqslant 1}(n+t-1)|.$$

In fact, we have the following theorem.

THEOREM 3. *For* $t\leqslant 0$, $f^{*(1-t)}$ *is a bijection, with inverse* $g^{*(1-t)}$, *from the set of partitions with minimum rank* t *to the set of partitions with all ranks positive. Furthermore, when applied to a partition* π, $f^{*(1-t)}$ *decreases the weight of* π *by* $-t+1$, *decreases the number of parts of* π *by* $-t+1$, *and preserves the size of the Durfee rectangle of* π.

Combining Theorems 2 and 3, we have the following result, which establishes (4) of Section 1.

COROLLARY 1. *For* $t\leqslant 0$, $G\circ f^{*(1-t)}$ *is a bijection,*

$$R_{=t}(n) \to P_1(n-t-1),$$

with inverse $g^{*(1-t)}\circ F$.

Using this we now construct a bijection for identity (3) of Section 1, mapping partitions of n with all ranks at least t to partitions of n with no part "$2-t$." Define h_t for $\pi\in R_{\geqslant t}(n)$ as follows.

$h_t(\pi)$:	(*given* $\pi\in R_{\geqslant t}(n)$)
Let s be the minimum rank of π;	(*Note* $s\geqslant t$.)
If $s\geqslant 1$ then $s\leftarrow 1$.	
i. Let $\sigma = G\circ f^{*(1-s)}(\pi)$	(*Then* $\sigma\in P_1(n+s-1)$ *by Corollary* 1.)
ii. Add $1-s$ copies of part "1" to σ.	(*Now* $\sigma\in P(n)$ *with exactly* $1-s$ *ones.*)
iii. Replace every occurrence of part "$2-t$" in σ by $2-t$ parts of size 1.	
The result is $h_t(\pi)$.	

$R_{\geq 0}(7)$		$P_2(7)$
(7)	$\longrightarrow$	(7)
(6,1)	$\longrightarrow$	(5,1,1)
(5,2)	$\longrightarrow$	(6,1)
(5,1,1)	$\longrightarrow$	(3,1,1,1,1)
(4,3)	$\longrightarrow$	(4,3)
(4,2,1)	$\longrightarrow$	(4,1,1,1)
(4,1,1,1)	$\longrightarrow$	(1,1,1,1,1,1,1)
(3,3,1)	$\longrightarrow$	(3,3,1)

FIG. 5. The bijection $h_0\colon R_{\geqslant 0}(7) \to P_2(7)$.

Note that after step (iii), $h_t(\pi) = \sigma \in P_{2-t}(n)$ and the number of parts "1" in $h_t(\pi)$ is congruent to $(1-s)$ modulo $(2-t)$, since $0 \leqslant 1-s < 2-t$.)

THEOREM 4. *As defined above, h_t is a bijection*

$$h_t\colon R_{\geqslant t}(n) \to P_{2-t}(n).$$

Proof. Clearly h_t maps $R_{\geqslant t}(n)$ to $P_{2-t}(n)$. Given $\alpha \in P_{2-t}(n)$, h_t^{-1} is defined as follows: Let z be the number of ones in α and write

$$z = a(2-t) + b,$$

where $0 \leqslant b < 2-t$. Replace $a(2-t)$ of the parts "1" in α by a parts "$2-t$." Delete the remaining b of the parts "1." Now apply $(G \circ f^{*(1-b)})^{-1} = (f^{*(1-b)})^{-1} \circ G^{-1} = g^{*(1-b)} \circ F$ to get a partition in $R_{=1-b}(n)$ which is a subset of $R_{\geqslant t}(n)$ since $1-b \geqslant t$. ∎

The bijection $h_0\colon R_{\geqslant 0}(7) \to P_2(7)$ is illustrated in Fig. 5.

4. GENERATING FUNCTIONS

The results of Sections 2 and 3 can be rephrased in terms of generating function identities. We let $p_1(d, k)$, $r_{\geqslant t}(d, k)$, $r_{=t}(d, k)$ denote the set of partitions with k parts and with Durfee rectangle size $d \times (d+1)$ in, respectively, P_1, $R_{\geqslant t}$, and $R_{=t}$. When we fix only the Durfee rectangle, we use $p_1(d)$, $r_{\geqslant t}(d)$, $r_{=t}(d)$, and when only the number of parts is fixed: $\bar{p}_1(k)$, $\bar{r}_{\geqslant t}(k)$, $\bar{r}_{=t}(k)$.

THEOREM 5. *The following results hold for $t \leqslant 0$.*

Fixing both the size of the Durfee rectangle, d, and the number of parts, k:

$$\sum_{\pi \in p_1(d,k)} q^{|\pi|} = \frac{q^{d(d+1)} q^{2(k-d)}}{(q)_d} \begin{bmatrix} k-1 \\ d-1 \end{bmatrix}_q$$

$$= \sum_{\lambda \in r_{\geqslant 1}(d,k)} q^{|\lambda|} = q^{t-1} \sum_{\sigma \in r_{=t}(d,k-t+1)} q^{|\sigma|}. \tag{5}$$

Summing (5) *over k, fixing only the size of the Durfee rectangle*:

$$\sum_{\pi \in p_1(d)} q^{|\pi|} = \frac{q^{d(d+1)}(1-q)}{(q)_{d+1}\,(q)_d} = \sum_{\lambda \in r_{\geqslant 1}(d)} q^{|\lambda|} = q^{t-1} \sum_{\sigma \in r_{=t}(d)} q^{|\sigma|}. \tag{6}$$

Summing (5) *over d, fixing only the number of parts*:

$$\sum_{\pi \in \bar{p}_1(k)} q^{|\pi|} = \frac{q^{2k}}{(q)_k} = \sum_{\lambda \in \bar{r}_{\geqslant 1}(k)} q^{|\lambda|} = q^{t-1} \sum_{\sigma \in \bar{r}_{=t}(k-t+1)} q^{|\sigma|}. \tag{7}$$

Summing (5) *over d and k, so that partitions are otherwise unrestricted*:

$$\sum_{\pi \in P_1} q^{|\pi|} = \frac{1-q}{(q)_\infty} = \sum_{\lambda \in R_{\geqslant 1}} q^{|\lambda|} = q^{t-1} \sum_{\sigma \in R_{=t}} q^{|\sigma|}. \tag{8}$$

Proof. We use the well-known generating functions for ordinary partitions:

$(q)_k^{-1}$ for partitions with at most k parts,

$\begin{bmatrix} m \\ k \end{bmatrix}_q$ for partitions with at most k parts and largest part at

most $m-k$, and $(q)_\infty^{-1}$ for $P(n)$,

where $(q)_k = (1-q)(1-q^2)\cdots(1-q^k)$; $[{m \atop k}]_q = (q)_m\,(q)_k^{-1}\,(q)_{m-k}^{-1}$; and $(q)_\infty = (1-q)(1-q^2)\cdots$.

In each of the identities (5)–(7), the first equality follows from Fig. 6 (a)–(c). In (8), the first equality follows from (1) of Section 1 with $b=1$.

The second equalities in (5)–(8) follow from Theorem 2 and the last equalities follow from Theorem 3. ∎

The first two equalities in (8) appear explicitly in [And93] and [RA95] and, as mentioned earlier, also follow as a special case of Andrews and

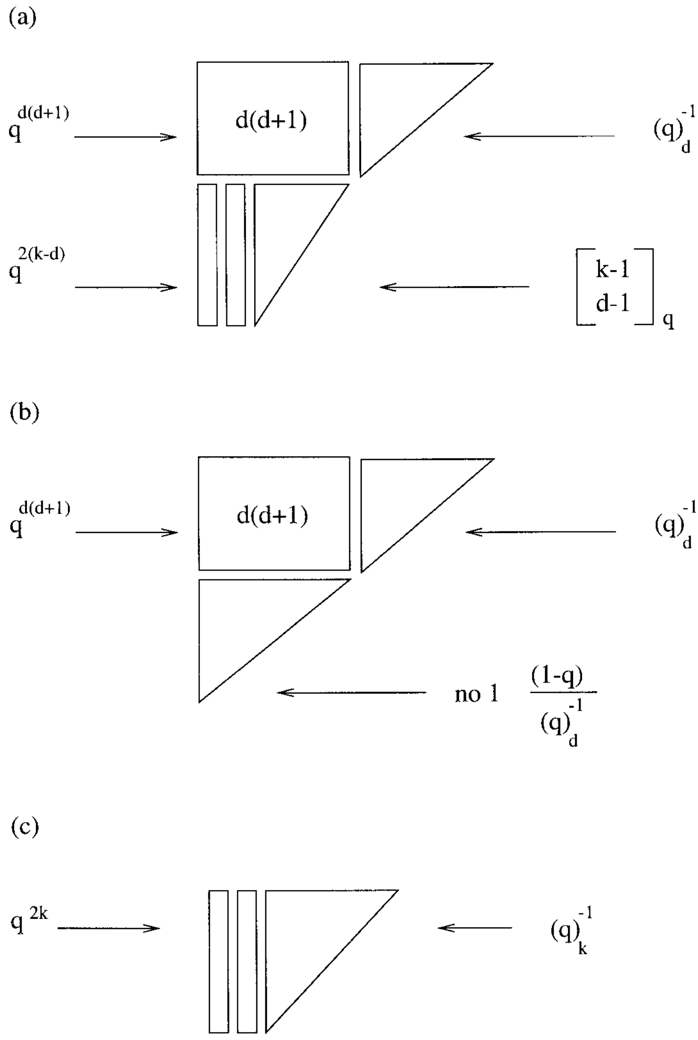

FIG. 6. Computing the generating function for partitions with no "1" from the Ferrers diagram. (a) Fix d, k; (b) fix only d; (c) fix only k.

Bressoud's Theorem 1 in Section 1. Note that for partitions in $R_{\geqslant 1}(n)$, the Durfee square and the Durfee rectangle are the same size. As a result, the second equality in 6 is the same as the one in [RA95] which uses MacMahon's generating function for plane partitions with bounded part size. Connections with plane partitions are discussed in the next section.

5. TWO-ROWED PLANE PARTITIONS

Our breakthrough in the search for a bijective proof of the identity $R_{\geqslant 1}(n) = P_1(n)$ came when we found a result of Cheema and Gordon on two-rowed plane partitions.

An *r-rowed plane partition* of n is an array of non-negative integers

$$\begin{array}{cccc} a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & a_{22} & a_{23} & \cdots \\ a_{31} & a_{32} & a_{33} & \cdots \\ \vdots & & & \\ a_{r1} & a_{r2} & a_{r3} & \cdots \end{array}$$

where $\sum_{i,j} a_{ij} = n$ and rows and columns are non-increasing.

We can regard a two-rowed plane partition of n as a pair of partitions (σ, τ), where $\sigma = \sigma_1, \sigma_2, ..., \tau = \tau_1, \tau_2, ..., |\sigma| + |\tau| = n$, and $\sigma_i \geqslant \tau_i$, $i = 1, 2, ...$.

Let $T(n)$ be the set of 2-rowed plane partitions of n which we will regard as pairs of partitions satisfying the constraints above. Let $S(n)$ be the set of pairs of partitions (α, β) satisfying:

— $|\alpha| + |\beta| = n$ and

— α has no part of size one.

In [CG64], Cheema and Gordon gave a bijection

$$\Theta\colon S(n) \to T(n).$$

We observed that by "pulling out" the Durfee rectangle of a partition $\pi \in P_1(n)$ and applying the Cheema–Gordon bijection to the pair of partitions remaining to the east and south of the rectangle, we could extend Θ to a bijection $\Theta^*\colon P_1(n) \to R_{\geqslant 1}(n)$. The idea is sketched below.

For a partition $\pi \in P(n)$ with Durfee square of size d and Durfee rectangle of size d^*, let σ be the partition $\sigma = (\pi_1 - d^* - 1, ..., \pi_d - d^* - 1)$ and τ the partition $\tau = (\pi'_1 - d^*, ..., \pi'_d - d^*)$. Represent π as the triple (d^*, σ, τ).

If $\pi \in P_1(n)$, then π contains no "1," so the conjugate τ' of τ contains no "1," and therefore $(\sigma', \tau') \in S(n - d^*(d^* + 1))$. On the other hand, if $\pi \in R_{\geqslant 1}(n)$, then since all ranks of π are positive, $(\sigma, \tau) \in T(n - d^*(d^* + 1))$. Since the Ferrers diagram of τ "fits inside" the Ferrers diagram of σ, we also have $(\sigma', \tau') \in T(n - d^*(d^* + 1))$.

Now extend the Cheema–Gordon bijection, Θ, to $\Theta^*\colon P_1(n) \to R_{\geqslant 1}(n)$ as follows. For $\pi = (d^*, \sigma, \tau) \in P_1(n)$, let $(\alpha, \beta) = \Theta((\sigma', \tau'))$. Define $\Theta^*(\pi) = \Theta^*((d^*, \sigma, \tau)) = (d^*, \alpha', \beta')$. It can be shown that $\Theta^*(\pi) \in R_{\geqslant 1}(n)$ and that Θ^* is a bijection.

It can be further shown that $\Theta^* = F$, the mapping described in Section 2. This is not surprising since we devised F as an alternative formulation of Θ^* which decomposes Θ^* into a sequence of basic steps, f, each of whose effect can be analyzed and altered to produce bijective proofs for more general versions of the identity.

We note that the mapping F also results in a simple bijective proof of the following formula which was generalized by Bender and Knuth in [BK72].

COROLLARY 2. *The number $|T(n, d, l)|$ of two-rowed plane partitions with parts of size at most d and exactly l parts in the second row is the coefficient of q^n in*:

$$\frac{q^{2l}}{(q)_d} \begin{bmatrix} d+l-1 \\ d-1 \end{bmatrix}_q \tag{9}$$

Proof. The mapping $(\sigma, \tau) \to (d, \sigma', \tau')$ gives a bijection between plane partitions in $T(n, d, l)$ and partitions in $R_{\geqslant 1}(n + d(d+1))$ with Durfee rectangle size d and exactly $d + l$ parts. The generating function for this second set is just $q^{-d(d+1)}$ times the generating function for $r_{\geqslant 1}(d, d+l)$, which is given by (5) of Theorem 5; it reduces to (9). ∎

6. DIFFERENCE OF SUCCESSIVE GAUSSIAN POLYNOMIALS

There are several interpretations of the difference of successive Gaussian polynomials, $[{n \atop k}]_q - [{n \atop k-1}]_q$, including those in [But87], [And93], and [Fis95]. We make just a few remarks here to relate our work to these. For some of the many interesting properties of these polynomials, their differences, and generalizations, see [But87, But90, Fis95].

Let $L[n, k]$ be the set of partitions whose Ferrers diagram lies in a $k \times (n-k)$ box. The number of such partitions is $\binom{n}{k}$. Let $L[m; n, k]$ be the partitions in $L[n, k]$ of weight m. The generating function for $L[m; n, k]$ is

$$\sum_{n \geqslant 0} L[m; n, k]\, q^m = \begin{bmatrix} n \\ k \end{bmatrix}_q$$

By Lemma 4, when $n \geqslant 2k$, g^* applied to a partition in $L[n, k]$ gives a partition in $L[n, k-1]$ and

$$g^*: L[n, k-1] \to L[n, k]$$

is an injection. Furthermore, the nonempty partitions in $L[n, k]$ not in the image of g^* are exactly those also in $R_{\geqslant 1}$. Letting $R_{\geqslant 1}[n, k]$ denote those partition in $L[n, k]$ with all ranks positive, we have

$$|R_{\geqslant 1}[n, k]| + 1 = |L[n, k]| - |L[n, k-1]| = \binom{n}{k} - \binom{n}{k-1} \tag{10}$$

and, since g^* increases the weight of a partition by 1, we have the following.

THEOREM 6.

$$\begin{bmatrix} n \\ k \end{bmatrix}_q - q \begin{bmatrix} n \\ k-1 \end{bmatrix}_q = \sum_{\lambda \in R_{\geqslant 1}[n, k]} q^{|\lambda|} + 1 \tag{11}$$

In [And93], Andrews shows, using a result from [And71], that

$$q^{-k} \left(\begin{bmatrix} n \\ k \end{bmatrix}_q - \begin{bmatrix} n \\ k-1 \end{bmatrix}_q \right) = \sum_{\pi \in A[n, k]} q^{|\pi|} \tag{12}$$

where $A[n, k]$ is the set of partitions in $L[n, k]$ with all ranks smaller than $n - 2k$. Thus, $A[n, k]$ is also counted by (10). We can establish a bijection between $A[n, k]$ and $R_{\geqslant 1}[n, k]$ with an idea used by Fishel in [Fis95] as follows.

Let $\pi \in R_{\geqslant 1}(n)$. For $1 \leqslant i \leqslant d$, replace π_i by $n - k - \pi_i + d$; reorder rows into nondecreasing order. For $1 \leqslant i \leqslant d$, replace π'_i by $k - \pi'_i + d$; reorder columns into nondecreasing order. The net result is that rank vector $[r_1, ..., r_d]$ becomes $[n - 2k - r_d, ..., n - 2k - r_1]$, giving a partition in $A[n, k]$. It is easy to check that this is a bijection which is, in fact, its own inverse.

We also mention the result of Fishel in [Fis95] that

$$\left(\begin{bmatrix} n \\ k \end{bmatrix}_q - \begin{bmatrix} n \\ k-1 \end{bmatrix}_q \right) = \sum_{\pi \in Q[n, k]} q^{|\pi|} \tag{13}$$

where $Q[n, k]$ is the set of all partitions π in $L[n, k]$ satisfying $\pi_1 \geqslant k$, $\pi'_d = d$, and for $i = 1, ..., d-1$, $\pi_{i+1} \geqslant \pi'_i$ (where d is the size of the Durfee square of π). Fishel exhibits a bijection between $Q[n, k]$ and $A[n, k]$. Below we define a bijection between $Q[n, k]$ and $R_{\geqslant 1}[n, k]$. We note that neither of our two bijections nor the one of Fishel is weight-preserving.

Define ϕ: $Q[n, k] \to R_{\geqslant 1}[n, k]$ by the following procedure.

$\phi(\pi)$:

if $\pi_1 = n - k$

then delete a part "$n - k$" from π;

Otherwise add a part "k" to π'.

Clearly, $\phi(\pi) \in L[n, k]$. To show $\phi(\pi) \in R_{\geqslant 1}[n, k]$, we must show all ranks of $\phi(\pi)$ are positive. If $\pi_1 = n-k$, then since $\pi'_d = d$, $\phi(\pi)$ has Durfee square size $d-1$ and rank vector $[r_1, ..., r_{d-1}]$ where for $1 \leqslant i \leqslant d-1$, $r_i = \pi_{i+1} - \pi'_i + 1 \geqslant 1$. Otherwise, $\pi_1 < n-k$ and $\phi(\pi)$ has Durfee square size d and rank vector $[r_1, ..., r_d]$ where $r_1 = \pi_1 + 1 - k \geqslant 1$ and for $1 \leqslant i \leqslant d-1$, $r_{i+1} = \pi_{i+1} + 1 - \pi'_i \geqslant 1$.

Now, define ϕ^{-1}: $R_{\geqslant 1}[n, k] \to Q[n, k]$ by the following procedure.

$\phi^{-1}(\pi)$:

if $\pi'_1 = k$

then delete a part "k" from π';

Otherwise add a part "$n-k$" to π.

To show ϕ is onto, we show that for $\pi \in R[n, k]$, $\phi^{-1}(\pi) \in Q[n, k]$ and $\phi\phi^{-1}(\pi) = \pi$. It is then easy to check that also $\phi^{-1}\phi$ is the identity on $Q[n, k]$.

Let $\pi \in R_{\geqslant 1}[n, k]$ with rank vector $[r_1, ..., r_d]$ where $r_r \geqslant 1$ for $1 \leqslant i \leqslant d$ and let $\sigma = \phi^{-1}(\pi)$. If $\pi'_1 = k$, then σ has Durfee square size d and $\sigma'_1 = k$, $\sigma'_d = \pi'_{d+1} = d$, and for $1 \leqslant i \leqslant d-1$,

$$\sigma_{i+1} - \sigma'_i = (\pi_{i+1} - 1) - \pi'_{i+1} = r_{i+1} - 1 \geqslant 0$$

Thus, $\sigma \in Q[n, k]$ and, since $\sigma_1 < n-k$, $\phi(\sigma) = \pi$, otherwise, $\pi'_1 < k$ and σ has Durfee square size $d+1$ and $\sigma'_1 = \pi'_1 + 1 \leqslant k$, $\sigma'_{d+1} = \pi'_{d+1} = d+1$, and, for $1 \leqslant i \leqslant d$,

$$\sigma_{i+1} - \sigma'_i = \pi_i - (\pi'_i + 1) = r_i - 1 \geqslant 0$$

Thus, $\sigma \in Q[n, k]$ and, since $\sigma_1 = n-k$, $\phi(\sigma) = \pi$.

7. THE RESULT OF BURGE

In [Bur81], Burge defines a set $C_{M,r}(n)$ which he proves bijectively has the same size as the set $A_{M,r}(n)$ defined in Theorem 1 of Section 1. $C_{M,r}(n)$ is the set of all partitions $(\pi_1, ..., \pi_l)$ of n which have at most $r-1$ parts of size one and which satisfy $\pi_j - \pi_{j+(M-3)/2} \geqslant 2$ for $1 \leqslant j \leqslant l-(M-3)/2$.

As noted in Section 1, when $r = 2-t$ and $M = n+r+1$, $A_{M,r}(n) = A_{n-t+3,2-t}(n) = R_{\geqslant 1}(n)$. In this case, $C_{M,r}(n) = C_{n-t+3,2-t}(n)$ is the set of partitions of n with at most $1-t$ ones. To see this, note that in order for the condition $\pi_j - \pi_{j+(M-3)/2} \geqslant 2$ to be violated, π must have at least

$P_1(10)$		$R_{\geq 1}(10)$
(10)	$\longrightarrow$	(8,1,1)
(8,2)	$\longrightarrow$	(7,3)
(7,3)	$\longrightarrow$	(6,4)
(6,4)	$\longrightarrow$	(5,5)
(6,2,2)	$\longrightarrow$	(4,4,2)
(5,5)	$\longrightarrow$	(9,1)
(5,3,2)	$\longrightarrow$	(5,4,1)
(4,4,2)	$\longrightarrow$	(6,3,1)
(4,3,3)	$\longrightarrow$	(7,1,1,1)
(4,2,2,2)	$\longrightarrow$	(5,3,1,1)
(3,3,2,2)	$\longrightarrow$	(10)
(2,2,2,2)	$\longrightarrow$	(6,1,1,1,1)

FIG. 7. The bijection of Burge $P_1(10) \rightarrow R_{\geqslant 1}(10)$.

$(M-1)/2$ parts. However, if $M=n+r+1$ and if π has at most $r-1$ ones, then the number of parts of π is bounded above by

$$\frac{n-(r-1)}{2}+(r-1)=\frac{n+r-1}{2}<\frac{n+r}{2}=\frac{M-1}{2}$$

In particular, for $r=1$ and $M=n+2$, since $A_{n+2,1}(n)=R_{\geqslant 1}(n)$ and $C_{n+2,1}(n)=P_1(n)$, it follows from Burge's result that $P_1(n)=R_{\geqslant 1}(n)$, giving another bijection between these two sets. We note that this one-to-one correspondence (illustrated in Fig. 7) is not the same as our bijection (illustrated in Fig. 4) and, for example, it does not preserve the Durfee square size or the number of parts.

For $t<1$, and $r=2-t$, $M=n+r+1$, we can use part of the mapping h_t of Theorem 4 to show that the number $|C_{n-t+3,2-t}(n)|$ of partitions of n with at most $1-t$ parts of size one is $|P_{2-t}(n)|$. (Simply replace every

$R_{\geq 0}(7)=A_{10,2}(7)$		$C_{10,2}(7)$		$P_2(7)$
(7)	$\longrightarrow$	(3,2,2)	$\longrightarrow$	(3,1,1,1,1)
(6,1)	$\longrightarrow$	(7)	$\longrightarrow$	(7)
(5,2)	$\longrightarrow$	(6,1)	$\longrightarrow$	(6,1)
(5,1,1)	$\longrightarrow$	(4,3)	$\longrightarrow$	(4,3)
(4,3)	$\longrightarrow$	(5,2)	$\longrightarrow$	(5,1,1)
(4,2,1)	$\longrightarrow$	(3,3,1)	$\longrightarrow$	(3,3,1)
(4,1,1,1)	$\longrightarrow$	(2,2,2,1)	$\longrightarrow$	(1,1,1,1,1,1,1)
(3,3,1)	$\longrightarrow$	(4,2,1)	$\longrightarrow$	(4,1,1,1)

FIG. 8. The bijection of Burge $R_{\geqslant 0}(7) \rightarrow P_2(7)$.

part "$2-t$" by $2-t$ parts "1.") Combining this with the bijection of Burge shows bijectively that

$$|R_{\geqslant t}(n)| = |A_{n-t+3,\,2-t}(n)| = |C_{n-t+3,\,2-t}(n)| = |P_{2-t}(n)|$$

Again, this bijection is different from our bijection h_t in Section 3. For example, when $n=7$ and $t=0$, compare our one-to-one correspondence in Fig. 5 with that of Burge in Fig. 8.

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